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Nilpotency in Classical Groups over a Field of Characteristic 2

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1. Introduction

1.1. Let k be a field, possibly of characteristic 2. Fix a number $e \in \{0, 1\}$. We define a *form space* to be a finite dimensional vector space V over k equipped with a bilinear form β: $V \times V \to k$ and a quadratic form α: $V \to k$ satisfying

(a) if $e=1$ then $\alpha(v)=\beta(v,v)=0$ for all $v \in V$.
(b) if $e=0$ then $\beta(v,w)=\alpha(v+w)-\alpha(v)-\alpha(w)$ for all $v, w \in V$.

Note that $\beta(w,v)=(-1)^e \beta(v,w)$ in either case.

Let V be a form space. If W is a subspace of V we put $W^\perp = \{v \in V \mid \beta(v, W) = 0\}$. The form space V is called *non-defective* if $V^\perp = 0$. It is called *non-degenerate* if $\dim(V^\perp) \leqq 1$ and $\alpha(v) \neq 0$ for all non-zero $v \in V^\perp$.

1.2. Assume that V is non-degenerate. V is defective if and only if $\mathrm{char}(k)=2$ and $e=0$ and $\dim(V)$ is odd. If $e=1$ then $\dim(V)$ is even.

Let $G=G(V)$ be the algebraic group in the sense of [1] of the automorphisms $g \in Gl(V)$ which leave the forms β and α invariant. Let $\mathfrak{g}=\mathfrak{g}(V)$ be its Lie algebra. If $e=1$ then G is the *symplectic* group $Sp(V)$, so it is connected and semi-simple. If $e=0$ then G is the *orthogonal* group $O(V)$, which is reductive. $O(V)$ has two components if V is non-defective. $O(V)$ is connected if V is defective.

1.3. Our object is to describe the conjugacy classes of the unipotent elements of G and of the nilpotent elements of $\mathfrak{g}$ under the adjoint action of G given by

$$\mathrm{ad}(g)\,x = g x g^{-1} \quad (g \in G,\ x \in G \text{ or } x \in \mathfrak{g}).$$

Fix a number $c \in \{0, 1\}$. If $c=1$ let $u \in G$ be unipotent and put $T=u-1$. If $c=0$ let $T \in \mathfrak{g}$ be nilpotent. In both cases we have a nilpotent endomorphism T of V and we are interested in the endomorphism $c+T$.

By the theory of Jordan normal forms there is a *partition* $(m_1, \ldots, m_r)$, that is a finite sequence of integers $m_1 \geqq \ldots \geqq m_r \geqq 1$, and a family of vectors $v_1, \ldots, v_r$ such that the vectors $T^a v_i$ with $0 \leqq a \leqq m_i - 1$ form a basis of V. The family

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$\{v_1, \ldots, v_r\}$ is called a *Jordan basis*. The partition is uniquely determined by T. We write $p(V, T) = (m_1, \ldots, m_r)$.

For $n \geqq 0$ we define the quadratic forms α_n: $V \to k$ by $\alpha_n(v) = \alpha(T^n v)$ if $e=0$, and $\alpha_n(v) = \beta(T^{n+1} v, T^n v)$ if $e=1$. The *index function* $\chi(V, T)$: $\mathbb{N} \to \mathbb{Z}$ is defined by

$$\chi(V, T)(m) = \min\{n \geqq 0 \mid T^m v = 0 \Rightarrow \alpha_n(v) = 0\}.$$

Assume that the field k is quadratically closed. Then a complete solution of the classification problem is obtained in Sect. 3. The main point is

Theorem (3.8). *Endomorphisms $c+T$ and $c+T'$ as above are conjugate under the adjoint action of G if and only if $p(V, T) = p(V, T')$ and $\chi(V, T) = \chi(V, T')$.*

1.4. The centralizer $Z = Z(V, T)$ and the infinitesimal centralizer $\mathfrak{z} = \mathfrak{z}(V, T)$ of T are defined by $Z = \{g \in G \mid g\,T = Tg\}$ and $\mathfrak{z} = \{X \in \mathfrak{g} \mid XT = TX\}$. It is clear that $\mathfrak{z}$ contains the Lie algebra of Z. In Sect. 4 we calculate the dimensions.

Theorem (4.4). *Let $p(V, T) = (m_1, \ldots, m_r)$ and $\chi(V, T) = \chi$.*

(a) $\dim(Z) = \sum (i\, m_i - \chi(m_i))$.
(b) $\dim(\mathfrak{z}) = \sum (i m_i - [\frac{1}{2}(m_i + 1 - e)]$, *if* $\operatorname{char}(k) \neq 2$, *or* $e=0$.
(c) $\dim(\mathfrak{z}) = \sum (i\, m_i - [\frac{1}{2}(m_i - 1)])$, *if* $\operatorname{char}(k) = 2$, $e = c = 1$.
(d) $\dim(\mathfrak{z}) = \sum (i\, m_i)$, *if* $\operatorname{char}(k) = 2$, $e=1$, $c=0$.

Here $[x]$ denotes the largest integer $\leqq x$. If $\operatorname{char}(k) \neq 2$ it follows that $\dim(\mathfrak{z}) = \dim(Z)$, as is well known, cf. [9] I§5.

1.5. Assume that G is not connected, or equivalently that $e=0$ and that V is non-defective. The identity component G^0 of G is a normal subgroup of index two. In Sect. 5 we treat the following two questions: a) Which unipotent conjugacy classes of G belong to G^0? b) If $\mathcal{O}$ is a G-orbit of unipotent elements of G^0, or of nilpotent elements of $\mathfrak{g}$, is $\mathcal{O}$ a single G^0-orbit or does it split into two G^0-orbits?

1.6. In Sect. 6 we give some tables of orbits with a comparison between the three cases: I $\operatorname{char}(k) \neq 2$, II $\operatorname{char}(k) = 2$ and $c=1$, III $\operatorname{char}(k) = 2$ and $c=0$. Putting $n = \dim(V)$ we write $G = Sp(n)$ if $e=1$, and $G^0 = SO(n)$ if $e=0$. For the case that $\operatorname{char}(k) \neq 2$ the tables may be compared with those given in [4]. There the corresponding weighted Dynkin-diagrams are given and the inclusion relations between the closures of the orbits.

1.7. *Historical Remarks.* For the cases with $\operatorname{char}(k) \neq 2$ all results are classical, cf. [7] and [3]. The best reference text is in [9]. Formula 1.4(b) is equivalent to [4] 3.8(b). For the case that $\operatorname{char}(k) = 2$ and $c=1$ most of our results are equivalent to results stated without proof in [6]. One may also consult [10]. My interest in this field was revived by discussions with R. Groszer and others in Bonn, 1977. I have used a generalisation of the method of [9].

2. Translation into Commutative Algebra

2.1. Fix $e \in \{0, 1\}$. Let V be a form space. Generalising 1.2, we define $G = G(V)$ to be the group of the elements $g \in Gl(V)$ satisfying

(a) $\beta(g v, g w)=\beta(v, w)$ for all $v, w \in V$.
(b) $\alpha(g v)=\alpha(v)$ for all $v \in V$.
(c) $\det(g)^2=1$.

The Lie algebra $\mathfrak{g}=\mathfrak{g}(V)$ consists of the elements $X \in \mathfrak{gl}(V)$ satisfying

(a′) $\beta(X v, w)+\beta(v, X w)=0$ for all $v, w \in V$.
(b′) If $e=0$ then $\beta(X v, v)=0$ for all $v \in V$.
(c′) trace $(X)=0$.

If $e=1$ the conditions (b) and (b′) are trivial. If $\mathrm{char}(k) \neq 2$ then (a) $\Rightarrow$ (b) and (a′) $\Rightarrow$ (b′). If $e=0$ then (b) $\Rightarrow$ (a) and (b′) $\Rightarrow$ (a′). If V is non-degenerate then (a) +(b) $\Rightarrow$ (c) and (a′)+(b′) $\Rightarrow$ (c′).

Fix $c \in \{0,1\}$. We define a *form module* to be a pair (V, T) where V is a form space and T is a nilpotent endomorphism such that $1+T \in G(V)$ if $c=1$ and that $T \in \mathfrak{g}(V)$ if $c=0$. This is equivalent to the conditions

(a″) $\beta(Tv, w)=\beta(v, -(1+c\,T)^{-1}\,Tw)$ for all $v, w \in V$.
(b″) If $e=0$ then $\beta(Tv, v)+c\,\alpha(Tv)=0$ for all $v \in V$.

If $\mathrm{char}(k) \neq 2$ then (a″) $\Rightarrow$ (b″). If $e=0$ then (b″) $\Rightarrow$ (a″).

The conjugacy problem of 1.3 is equivalent to the classification of the form modules (V, T) on a given non-degenerate form space V, cf. [9] IV 2.7.

2.2. Let $A=k[[t]]$ be the ring of formal power series in the indeterminate t. If V is an A-module with $\dim(V)<\infty$, the endomorphism T of V given by $Tv=t\,v$ is nilpotent. Conversely, if V is a vector space with a nilpotent endomorphism T we define $V=(V, T)$ as the A-module given by $(\sum a_n t^n)\,v=\sum (a_n t^n v)$. All form modules will be considered as A-modules in this way.

2.3. In view of 2.1 condition (a″) we define the algebra automorphism σ of A by

$$\sigma(\textstyle\sum a_n t^n)=\sum a_n(1+c\,t)^{-n}(-t)^n.$$

One verifies that σ is an involution. Let A_σ denote the subring of A consisting of the formal power series in $t\,\sigma(t)$. Clearly A_σ is contained in the ring of invariants A^σ. We have

$$t+\sigma(t)=-c\,t\,\sigma(t)$$
$$t^{n+2}+\sigma(t)^{n+2}=(t^{n+1}+\sigma(t)^{n+1})(t+\sigma(t))-(t^n+\sigma(t)^n)\,t\,\sigma(t).$$

It follows that $a+\sigma(a) \in A_\sigma$ and $a\,\sigma(a) \in A_\sigma$ for all $a \in A$. (In particular $A_\sigma=A^\sigma$ if $\mathrm{char}(k) \neq 2$. If $\mathrm{char}(k)=2$ and $c=0$ then $A_\sigma \neq A^\sigma=A$).

2.4. We define E to be the vector space of the linear functionals $u\colon A \to k$ with $\langle u, t^n A\rangle=0$ for some n. The functionals t^{-n}, $n \geqq 0$, given by $\langle t^{-n}, \sum a_i t^i\rangle=a_n$ form a countable basis of E. In [5] the space E is identified with the polynomial ring $k[t^{-1}]$. We consider E as an A-module by the definition $\langle a\,u, x\rangle=\langle u, a\,x\rangle$ for $a, x \in A$, $u \in E$. The involution σ of A induces an involution σ of E by $\langle\sigma(u), x\rangle =\langle u, \sigma(x)\rangle$. For $a \in A$, $u \in E$ we have $\sigma(a\,u)=\sigma(a)\,\sigma(u)$. One verifies that

$$\sigma(t^{-n})=(-1)^n(1+c\,t)^{n-1}\,t^{-n}.$$

The space E is equipped with a new basis u_n, $n \geqq 0$, given by

$$u_n = (-1)^{[\frac{1}{2}n]}(1+c\,t)^{[\frac{1}{2}(n-1)]}\,t^{-n}.$$

For $m, n \geqq 0$ we have

$$\langle u_n, t^{[\frac{1}{2}(m+1)]}\,\sigma(t)^{[\frac{1}{2}m]}\rangle = \delta_{m,n}.$$

Now E is the direct sum $E_0 \oplus E_1$ of the A_σ-submodules $E_i = \sum k\,u_{2n+i}$ for $i \in \{0,1\}$. The corresponding projections π_0 and π_1 satisfy

$$\pi_i(u) = \sum_n \langle u, t^{n+i}\,\sigma(t)^n\rangle\, u_{2n+i}.$$

We define $h = (2+c\,t)(1+c\,t)^{-1}$ in A, so that $\sigma(h) = 2+c\,t$.

2.5. **Lemma.** *For $u \in E$ we have $u + \sigma(u) = \sigma(h)\,\pi_0(u)$ and $u - \sigma(u) = \pi_1(h\,u)$.*

Proof. It is more easy to verify the equivalent formulas: $u \in E_1 \Rightarrow \sigma(u) = -u$ and $u \in E_0 \Rightarrow \sigma(u) = (1+c\,t)\,u$ and $\pi_0(\sigma(u)) = \pi_0(u)$ and $\pi_1(\sigma(u)) = \pi_1(-(1+c\,t)^{-1}\,u)$.

2.6. We define an *abstract form module* V to be an A-module V with $\dim(V) < \infty$, equipped with mappings $\varphi\colon V \times V \to E$ and $\psi\colon V \to E$ which satisfy the following axioms:

(1) The map $\varphi(?, w)$ is A-linear for every $w \in V$.
(2) $\varphi(w, v) = (-1)^e\,\sigma(\varphi(v, w))$ for all $v, w \in V$.
(3) $\psi(v) \in E_e$ for all $v \in V$.
(4) $\varphi(v, v) = \sigma(h)^{1-e}\,\psi(v)$ for all $v \in V$.
(5) $\psi(v+w) - \psi(v) - \psi(w) = \pi_e(h^e\,\varphi(v, w))$ for all $v, w \in V$.
(6) $\psi(a\,v) = a\,\sigma(a)\,\psi(v)$ for $v \in V$, $a \in A$.

Remark. If $e=1$ the axioms (5) and (6) follow from the other ones (use 2.5 for 5). If $\mathrm{char}(k) \neq 2$ then (3), (5) and (6) follow from (1), (2) and (4). Therefore ψ is not introduced in [9] IV 2.5.

2.7. **Proposition.** *If (V, φ, ψ) is an abstract form module then (V, β, α) given by $(*)$ is a form module. If (V, β, α) is a form module there is a unique abstract form module (V, φ, ψ) such that $(*)$ holds; it is given by $(**)$.*

$(*)$ $\beta(v, w) = \langle \varphi(v, w), 1\rangle$, $\alpha(v) = \langle \psi(v), 1\rangle$
$(**)$ $\varphi(v, w) = \sum \beta(t^n\,v, w)\,t^{-n}$, $\psi(v) = \sum \alpha_n(v)\,u_{2n+e}$
(see 1.3 for the definition of α_n)

Proof. Let (V, φ, ψ) be an abstract form module. Let β and α be given by $(*)$. It is easy to see that (V, β, α) is a form space, and that the nilpotent endormorphism T of V, cf. 2.2, satisfies condition 2.1(a″). If $e=0$ then $\langle \psi(v), t\,A_\sigma\rangle = 0$, so that

$$\begin{aligned}\beta(Tv, v) + c\,\alpha(Tv) &= \langle t\,\varphi(v, v) + c\,t\,\sigma(t)\,\psi(v), 1\rangle \\ &= \langle \psi(v), t((1+c\,t) + \sigma(1+c\,t))\rangle = 0.\end{aligned}$$

This proves condition 2.1(b″), so that (V, β, α) is a form module.

Conversely, let (V, β, α) be a form module. Assume that (V, φ, ψ) is a compatible structure of an abstract form module. We have

$$\begin{aligned}
&\langle \varphi(v,w), t^n \rangle = \langle t^n \varphi(v,w), 1 \rangle = \langle \varphi(t^n v, w), 1 \rangle = \beta(t^n v, w)\\
&e=0 \Rightarrow \langle \psi(v), t^n \sigma(t)^n \rangle = \langle \psi(t^n v), 1 \rangle = \alpha(t^n v) = \alpha_n(v)\\
&e=1 \Rightarrow \langle \psi(v), t^{n+1} \sigma(t)^n \rangle = \langle \varphi(t^{n+1} v, t^n v), 1 \rangle = \alpha_n(v).
\end{aligned}$$

Now (**) follows using axiom (3). This proves uniqueness. For the proof of the existence we use (**) as a definition. Now we have $\langle \varphi(v,w), a \rangle = \beta(a v, w)$ for $a \in A$, $v, w \in V$. This implies axiom (1). Condition 2.1(a″) implies that $\beta(a v, w) = \beta(v, \sigma(a) w)$ for all $a \in A$.

Now one proves axiom (2). Axiom (3) follows immediately from (**). Using the formulas in 2.4 for the projections π_0 and π_1 and the fact that $\pi_0 + \pi_1 = \mathrm{id}$, we obtain

$$\varphi(v,v) = \sum \beta(t^n v, t^n v)\, u_{2n} + \sum \beta(t^{n+1} v, t^n v)\, u_{2n+1}.$$

If $e=1$ then β is alternating so that $\varphi(v,v) = \psi(v)$. If $e=0$ it follows from 2.1(b″) that

$$\varphi(v,v) = \sum 2\alpha_n(v)\, u_{2n} - c \sum \alpha_{n+1}(v)\, u_{2n+1}.$$

Since $t u_{2n} = -u_{2n-1}$ and $t u_0 = 0$, we get $\varphi(v,v) = (2 + c t)\psi(v)$. This concludes the verification of axiom (4). In view of 2.6 remark we now assume that $e=0$. Axiom (5) follows from

$$\begin{aligned}
&\langle \psi(v+w) - \psi(v) - \psi(w), t^n \sigma(t)^n \rangle = \alpha(t^n(v+w)) - \alpha(t^n v) - \alpha(t^n w)\\
&\quad = \beta(t^n v, t^n w) = \langle \varphi(v,w), t^n \sigma(t)^n \rangle.
\end{aligned}$$

By 2.5 and axiom (4) we have $\psi(v) + \sigma\psi(v) = \varphi(v,v)$. It follows that

$$\langle \psi(v), t^n \sigma(t)^m + t^m \sigma(t)^n \rangle = \langle \varphi(v,v), t^n \sigma(t)^m \rangle = \beta(t^n v, t^m v).$$

Using this one proves $\langle \psi(v), a\sigma(a) \rangle = \alpha(a v)$ for every $a \in A$. It follows that

$$\langle a\sigma(a)\psi(v), t^n \sigma(t)^n \rangle = \alpha(t^n a v) = \langle \psi(a v), t^n \sigma(t)^n \rangle.$$

Since both sides of the equation are element of E_0 this proves that $a\sigma(a)\psi(v) = \psi(a v)$.

2.8. Let V be an A-module. We define $\mu(V)$ to be the minimal integer $m \geqq 0$ with $t^m V = 0$. If $t^m V \neq 0$ for all m we put $\mu(V) = +\infty$. For $v \in V$ we define $\mu(v) = \mu(A v)$. Assume $\dim(V) < \infty$. As in 1.3 there is a unique *partition* $p_V = (m_1, \ldots, m_r)$ and some family $v_1, \ldots, v_r$, called a *Jordan basis*, such that V is the direct sum $\sum_{i=1}^{r} A v_i$ and that $m_i = \mu(v_i)$ for $i = 1, \ldots, r$. For the uniqueness it is required that $m_1 \geqq \ldots \geqq m_r \geqq 1$. We shall meet systems with $\mu(v_r) = 0$.

Henceforward we use 2.7 to identify a form module (V, T) and the corresponding abstract form module V. One verifies that the *index function* $\chi_V = \chi(V, T)$ of a form module cf. 1.3 satisfies $\chi_V(m) = \max\{[\tfrac{1}{2}(\mu\psi(v) + 1)] \mid v \in V,\ t^m v = 0\}$.

3. Normal Forms

3.1. Let V be a form module. Every submodule of V is a form module in its own right. If W is a subspace of V then $W^\perp$ is defined in terms of β, cf. 1.1. If W is a submodule one verifies that $W^\perp$ consists of the elements $v\in V$ with $\varphi(v, W)=0$, so that $W^\perp$ is a submodule. Submodules W and W' are called *orthogonal* if $W'\subset W^\perp$. Then $\psi(w+w')=\psi(w)+\psi(w')$ for all $w\in W$, $w'\in W'$.

An *orthogonal decomposition* of V is an expression of V as a direct sum $V=\sum_{i=1}^{r} V_i$ of mutually orthogonal submodules V_i. The form module V is called *indecomposable* if $V\neq 0$ and for every orthogonal decomposition $V=V_1\oplus V_2$ we have $V_1=0$ or $V_2=0$. Every form module V has some orthogonal decomposition $V=\sum_{i=1}^{r} V_i$ in indecomposable submodules $V_1,\dots,V_r$. One verifies that V is non-degenerate if and only if every V_i is non-degenerate and at most one of them is defective.

3.2. **Proposition.** *Let V be a non-degenerate indecomposable form module. There exist v_1, $v_2\in V$ such that $V=A v_1\oplus A v_2$ and $\mu(v_1)\geqq\mu(v_2)$.*

For any such pair we put $m=\mu(v_1)$, $m'=\mu(v_2)$, $\Phi=\varphi(v_1, v_2)$ and $\Psi_i=\psi(v_i)$. One of the following conditions holds:

a) *$m'=0$, $\Phi=\Psi_2=0$. If* $\operatorname{char}(k)\neq 2$ *or $e=1$ then $\mu(\Psi_1)=m$. If* $\operatorname{char}(k)=2$ *and $e=0$ then $c=1$ and $\mu(\Psi_1)=m+1$.*

b) *$m'=\mu(\Phi)=m$. If* $\operatorname{char}(k)\neq 2$ *then $m\equiv e$* $(\operatorname{mod} 2)$. *If* $\operatorname{char}(k)\neq 2$ *or $e=1$ then $\mu(\Psi_i)\leqq m-1$. If* $\operatorname{char}(k)=2$, *$e=0$, $c=1$, then $\mu(\Psi_i)\leqq m$. If* $\operatorname{char}(k)=2$, *$e=c=0$ then $\mu(\Psi_i)\leqq 2m-1$.*

c) *$m'=\mu(\Phi)=m-1$;* $\operatorname{char}(k)=2$; *$e=0$; if $m\geqq 2$ then $c=0$; $\mu(\Psi_1)=2m-1>\mu(\Psi_2)$.*

Proof. a) Assume the existence of a non-zero vector $v_1\in V$ with $\mu\varphi(v_1, v_1)=\mu(v_1)$. Put $v_2=0$ and define m, m', Φ, Ψ_i as above. For $v\in V$ and $a\in A$ we have $v-av_1\in(Av_1)^\perp$ if and only if $a\varphi(v_1, v_1)=\varphi(v, v_1)$. Since $\mu\varphi(v, v_1)\leqq m$ there is $a\in A$ with $v-av_1\in(Av_1)^\perp$. This $a\in A$ is unique modulo $t^m A$. So $V=A v_1\oplus(A v_1)^\perp$. Since V is indecomposable it follows that $V=A v_1$. Now use 2.6(4).

b) Assume that $\mu\varphi(v, v)<\mu(v)$ for all non-zero $v\in V$. Assume the existence of non-zero vectors $v_1, v_2\in V$ with $\mu(v_1)=\mu\varphi(v_1, v_2)=\mu(v_2)$. Define m, Φ, Ψ_i as above. Put $A'=A/t^m A$ and $W=A v_1+A v_2$. Let $v\in V$. Write $\varphi(v_i, v_j)=a_{ij}t^{1-m}$ and $\varphi(v_i, v)=a_i t^{1-m}$ with a_{ij}, $a_j\in A'$. We have a_{11}, $a_{22}\in tA'$ and a_{12}, $a_{21}\notin tA'$. For x_1, $x_2\in A'$ we have

$$v-x_1 v_1-x_2 v_2\in W^\perp \Leftrightarrow \forall i\colon a_i=\sum_j a_{ij}x_j.$$

The righthand side has a unique solution. So $W=A v_1\oplus A v_2$ and $V=W\oplus W^\perp$. Since V is indecomposable we have $V=W$. Now use the above assumption and 2.6(4).

c) It remains to consider the case that $\mu(v)=\mu\varphi(v, w)=\mu(w)$ implies $v=w=0$. This implies that $\mu\varphi(v, w)<\mu(V)$ for all $v, w\in V$. Putting $m=\mu(V)$ we get

$t^{m-1} V \subset V^{\perp}$. Choose $v_1 \in V$ with $\mu(v_1)=m$ and hence $t^{m-1} v_1 \neq 0$. So V is defective. Since V is non-degenerate we have $\mathrm{char}(k)=2$, $e=0$, $V^{\perp}=k\, t^{m-1} v_1$ and $a(t^{m-1} v_1) \neq 0$. By 2.7 (**) it follows that $\mu\psi(v_1)=2m-1$. If $m=1$ it is clear that $V=k v_1$, and we are done. Assume that $m \geqq 2$. Then we have

$$\mu(c\, t\, \psi(v_1)) = \mu\, \varphi(v_1, v_1) \leqq m-1 \leqq \mu\, \psi(v_1) - 2.$$

This implies that $c=0$. So for all $v, w \in V$ we have $\varphi(v,v)=0$ and $\varphi(v,w)=\varphi(w,v)$. Since $t^{m-2} v_1 \notin V^{\perp}$ there is $v_2 \in V$ with $\mu\, \varphi(v_1, v_2)=m-1$. Replacing v_2 by $v_2 - \gamma v_1$ for some $\gamma \in k$, we may assume that $\mu(v_2)=m-1$. Put $W=A v_1 + A v_2$. With the method used in (b), one shows that $W=A v_1 \oplus A v_2$, that $V=W+W^{\perp}$ and that $V^{\perp}=W \cap W^{\perp}$. Suppose $V \neq W$. Then there exist $n \in \mathbb{N}$ and $w_1 \in W^{\perp}$ with $t^n(W^{\perp}) \subset V^{\perp}$ and $t^{n-1} w_1 \notin V^{\perp}$. There is $w_2 \in W^{\perp}$ with $\varphi(t^{n-1} w_1, w_2) \neq 0$. Replacing w_i by $w_i - \gamma_i t^{m-1-n} v_1$ for some $\gamma_i \in k$, we may assume that

$$\mu(w_1) = \mu\, \varphi(w_1, w_2) = \mu(w_2) = n$$

contradicting the preliminary assumption of c). This proves that $V=W$. Since $t^{m-1} v_2 = 0$ we have $\mu\psi(v_2) \leqq 2m-3$.

3.3. **Proposition.** *Let $m \in \mathbb{N}$, $m' \in \mathbb{N} \cup \{0\}$, $\Phi \in E$, $\Psi_1, \Psi_2 \in E_e$ be given satisfying the conditions in* 3.2(a) *or* (b) *or* (c). *Up to a canonical isomorphism there exists a unique form module $V=A v_1 \oplus A v_2$ with $m=\mu(v_1)$, $m'=\mu(v_2)$, $\Phi=\varphi(v_1, v_2)$, $\Psi_i = \psi(v_i)$. This form module V is indecomposable. In the cases* 3.2 (a) *and* (b) *it is non-defective. In case* (c) *it is defective and non-degenerate.*

Proof. The uniqueness follows from 2.6(1), (2), (4). The proof of the existence is a formal exercise, – use 2.5. In case (a) the A-module V is indecomposable. In the cases (b) and (c) one uses 3.2 to prove that the form module V is indecomposable. The remaining assertions are proved essentially in 3.2.

3.4. **Normalisation.** The field k is called *quadratically closed* if every quadratic equation $x^2 + a x + b = 0$ with $a, b \in k$ has a solution $x \in k$. Assume that the field k is quadratically closed. We normalise the base vectors v_1 and v_2 of the non-degenerate form modules of 3.2 in some steps.

(i) Every $y \in A_\sigma$ can be written $y = x\, \sigma(x)$ with $x \in A$. One proves this using an infinite product and the following formulas for $a \in k$ and $n \in \mathbb{N}$.

$$z = 1 + a\, t^n \sigma(t)^n \Rightarrow z\, \sigma(z) = 1 + 2\, a\, t^n \sigma(t)^n + a^2 t^{2n} \sigma(t)^{2n}$$
$$z = 1 + a\, t^n \sigma(t)^{n-1} \Rightarrow z\, \sigma(z) = 1 - c\, a\, t^n \sigma(t)^n + a^2 t^{2n-1} \sigma(t)^{2n-1}.$$

(ii) An immediate consequence of (i) is that in 3.2(a) we can choose v_1 such that the following holds. If $\mathrm{char}(k) \neq 2$ or $e=1$ then $\Psi_1 = u_{m-1}$. If $\mathrm{char}(k)=2$ and $e=0$ then $\Psi_1 = u_m$.

(iii) In 3.2(b) and (c) we may assume that $\mu(\Psi_1) \geqq \mu(\Psi_2)$. Assume that $\Psi_2 \neq 0$. Define

$$i = \max(\mu(h^e \Phi) - \mu(\Psi_2), \tfrac{1}{2}(\mu(\Psi_1) - \mu(\Psi_2))).$$

Note that i is a non-negative integer, and that $i \geqq 1$ in case 3.2(c). We may replace v_2 by $v_2' = a\,t^i\,v_1 + v_2$ for $a \in k$. We have

$$\psi(v_2') = \Psi_2 + a\,\pi_e(t^i\,h^e\,\Phi) + a^2\,t^i\,\sigma(t)^i\,\Psi_1$$
$$\mu(\Psi_2) = \max(\mu\,\pi_e(t^i\,h^e\,\Phi), \mu\,t^i\,\sigma(t)^i\,\Psi_1).$$

Since k is quadratically closed we can choose $a \in k$ such that $\mu(\psi(v_2')) < \mu(\Psi_2)$. It follows that we may assume that $\Psi_2 = 0$.

(iv) Assume that $\mu(\Psi_1) \leqq \mu(h^e\,\Phi)$ and $\Psi_2 = 0$. Choose $a \in A$ with $\Psi_1 = a\,h^e\,\Phi$. We may replace v_1 by $v_1' = v_1 - \sigma(a)\,v_2$ satisfying $\psi(v_1') = 0$. Multiplying v_2 by an invertible element of A we get $\Phi = u_{m-1}$, $\Psi_1 = \Psi_2 = 0$.

(v) Assume that $\mu(\Psi_1) > \mu(h^e\,\Phi)$ and $\Psi_2 = 0$ in 3.2(b) or (c). One verifies that $\mathrm{char}(k) = 2$ and $c = 0$. If $e = 1$ then $h^e\,\Phi = 0$ so that $\mu(\Psi_1) = 2\,l$ for some integer l with $0 < l < \frac{1}{2}m$. If $e = 0$ then $\mu(\Psi_1) = 2\,l - 1$ with $\frac{1}{2}(m+1) < l \leqq m$. In case 3.2(c) we have $l = m$. Using step (i) and multiplying v_1 and v_2 by invertible elements of A we obtain $\Psi_1 = u_{2l-2+e}$, $\Psi_2 = 0$ and $\Phi = t^{1-m}$ in case 3.2(b), $\Phi = t^{2-m}$ in 3.2(c).

Remark. If $\mathrm{char}(k) \neq 2$ the above normalisation of 3.2(b) works without the assumption that k is quadratically closed. If $\mathrm{char}(k) = 2$ and $c = 0$ the normalisation of 3.2(a) and (c) only requires that k is perfect.

3.5. In the rest of this paper we assume that the field k is quadratically closed. We define the functions $[m; l]\colon \mathbb{N} \to \mathbb{Z}$ by

$$[m; l](n) = \max\{0, \min\{n - m + l, l\}\}.$$

Now 3.4 implies

Proposition. *The indecomposable non-degenerate form modules V are $V(m)$, $W(m)$, $W_l(m)$, $D(m)$ with $m = \mu(V)$ and $\chi_V = [m; l]$, given in Table 1. Among these types only the types $D(m)$ are defective.*

Table 1.

$V = Av_1 \oplus Av_2$	char(k)	e	c	condition	$\mu(v_1)$	$\mu(v_2)$	$\psi(v_1)$	$\varphi(v_1, v_2)$	$\psi(v_2)$
$V(m)$	a	1	a	$m = 2l$					
	$\neq 2$	0	a	$m = 2l - 1$	m	0	u_{2l-2+e}	0	0
	2	0	1	$m = 2l - 2$					
$W(m)$	$\neq 2$	a	a	$m = 2l + e$					
	2	0	a	$l = [\frac{1}{2}(m+1)]$	m	m	0	t^{1-m}	0
	2	1	1	$l = [\frac{1}{2}(m-1)]$					
	2	1	0	$l = 0$					
$W_l(m)$	2	1	0	$0 < l < \frac{1}{2}m$	m	m	u_{2l-2+e}	t^{1-m}	0
	2	0	0	$\frac{1}{2}(m+1) < l \leqq m$					
$D(m)$	2	0	0	$m = l \geqq 2$	m	$m-1$	u_{2m-2}	t^{2-m}	0
	2	0	0	$m = l = 1$					

"a" means "arbitrary".

3.6. **Lemma.** *Let V be a form module. Let W be an indecomposable non-degenerate submodule of V. Let F be a non-degenerate form module with $p_W = p_F$ and $\chi_W < \chi_F \leqq \chi_V$. Then we have* $\mathrm{char}(k) = 2$ *and*

$$V = W \oplus W^\perp \cong F \oplus W^\perp.$$

Proof. By $\chi_W < \chi_F$ we mean that $\chi_W(n) \leqq \chi_F(n)$ for all $n \in \mathbb{N}$ and that $\chi_W(n) < \chi_F(n)$ for some $n \in \mathbb{N}$. Now it follows from 3.5 that $\mathrm{char}(k) = 2$, that $p_W = (m, m)$ for some $m \geqq 1$, that W is isomorphic to $W(m)$ or $W_l(m)$ and that F is isomorphic to $W_r(m)$ or an orthogonal sum $V(m) \oplus V(m)$. In particular W is non-defective so that $V = W \oplus W^\perp$.

We can choose a Jordan basis v_1, v_2 of W with $\psi(v_2) = 0$ and $\mu \varphi(v_1, v_2) = m$. We have $\chi_F = [m; r]$ for some r with $\chi_W(m) < r \leqq \chi_V(m)$. So there is an element $v' \in V$ with $t^m v' = 0$ and $\mu \psi(v') = 2r - 1 + e$. We may assume that $v' \in W^\perp$. Put $v_1' = v_1 + v'$ and $F' = A v_1' + A v_2$. One verifies that F' is non-defective and isomorphic to F as a form module. So we may assume that $F = F'$. Then V is the orthogonal direct sum $V = F \oplus F^\perp$.

Now it suffices to prove that the form modules $W^\perp$ and $F^\perp$ are isomorphic. The orthogonal projection P: $W^\perp \to F^\perp$ is a morphism of A-modules. Its kernel is $W^\perp \cap F = 0$. Since the dimensions of $W^\perp$ and $F^\perp$ are equal the mapping P is bijective. For $v \in W^\perp$ one verifies that $v - P v \in A v_2$. It follows that P: $W^\perp \to F^\perp$ is an isomorphism of form modules.

3.7. Let V be a non-degenerate form module. Up to isomorphisms and permutations there is a unique sequence $W(1), \ldots, W(r)$ of non-degenerate indecomposable form modules satisfying

a) $W(i)$ is non-defective for every $i < r$.
b) The partition p_V is the disjoint union of the partitions $p_{W(i)}$, $1 \leqq i \leqq r$.
c) $\chi_V(n) = \sup_i \chi_{W(i)}(n)$ for every $n \in \mathbb{N}$.
d) If $p_{W(i)} = (m_1, \ldots, m_s)$ and $1 \leqq j \leqq s$ then $\chi_{W(i)}(m_j) = \chi_V(m_j)$.

By 3.6 we have that V is isomorphic to the orthogonal direct sum $\sum_{i=1}^{r} W(i)$. This sum is called the *normal form* of V. Note that the isomorphism follows already from a), b), c). Condition d) is imposed to have uniqueness. Actually d) implies c). It follows from the existence of a normal form characterised by p_V and χ_V, that we have

3.8. **Theorem.** *Non-degenerate form modules V and W are isomorphic if and only if $p_V = p_W$ and $\chi_V = \chi_W$.*

Remark. The formulation in the end of 1.3 is slightly weaker, see the end of 2.1. If $\mathrm{char}(k) \neq 2$, non-degenerate form modules V and W are isomorphic if and only if $p_V = p_W$. This follows from 3.6, but it is a classical result, cf. [3] and [7].

3.9. We shall use a combinatorial formalisation of the above results. Let V be a non-degenerate form module with partition $p_V = (m_1, \ldots, m_r)$ and index function $\chi = \chi_V$. The exponent function n: $\mathbb{N} \to \mathbb{Z}$ is defined by

$$n(m) = \#\{i \mid m_i = m\}.$$

Here $\#$ denotes the cardinality of the set. Consider the finite set $I = \{m | n(m) \geqq 1\}$. By 3.5 and 3.7 we have

$$\chi = \sup_{m\in I}[m; \chi(m)].$$

It follows that the isomorphism class of V is determined by the *symbol*

$$S(V) = (p^{n(p)}_{\chi(p)}, q^{n(q)}_{\chi(q)}, \ldots, s^{n(s)}_{\chi(s)})$$

where $p > q > \ldots > s$ are the elements of I. A symbol S consisting of a finite set I with mappings $n\colon I \to \mathbb{N}$ and $\chi\colon I \to \mathbb{N} \cup \{0\}$ is the symbol of a non-degenerate form module if and only if it satifies the following rules:

a) For $i > j$ in I we have $\chi(i) \geqq \chi(j)$ and $i - \chi(i) \geqq j - \chi(j)$.
b) Assume $e = 1$ and $m \in I$. Then $0 \leqq \chi(m) \leqq \frac{1}{2}m$ and

$\frac{1}{2}(m-1) \leqq \chi(m) \leqq \frac{1}{2}m$ if $\operatorname{char}(k) \neq 2$;
$\frac{1}{2}(m-2) \leqq \chi(m) \leqq \frac{1}{2}m$ if $c = 1$;
$\chi(m) = \frac{1}{2}m$ if $n(m)$ is odd.

c) Assume $e = 0$ and $m \in I$. Then $\frac{1}{2}m \leqq \chi(m) \leqq m$ and
$\frac{1}{2}m \leqq \chi(m) \leqq \frac{1}{2}(m+1)$ if $\operatorname{char}(k) \neq 2$;
$\chi(m) = \frac{1}{2}(m+1)$ if $\operatorname{char}(k) \neq 2$ and $n(m)$ is odd;
$\frac{1}{2}m \leqq \chi(m) \leqq \frac{1}{2}(m+2)$ if $c = 1$;
$\chi(m) = \frac{1}{2}(m+2)$ if $\operatorname{char}(k) = 2$, $c = 1$, $m \geqq 2$ and $n(m)$ is odd;
$\chi(m) = m$ if $\operatorname{char}(k) = 2$, $c = 0$ and $n(m)$ is odd.

d) If $\operatorname{char}(k) = 2$ and $e = c = 0$ then $\{i \in I | n(i) \text{ odd}\} = \{m, m-1\} \cap \mathbb{N}$ for some $m \in \mathbb{Z}$.

4. Dimensions of Centralizers

4.1. Let V be a finite dimensional A-module. We consider the centralizers

$$\mathfrak{c}(V) = \{X \in \operatorname{End}(V) | XT = TX\} = \operatorname{End}_A(V)$$
$$C(V) = \{g \in Gl(V) | gT = Tg\}$$

If moreover V is a form module we have the centralizers

$$\mathfrak{z}(V) = \mathfrak{g}(V) \cap \mathfrak{c}(V), \qquad Z(V) = G(V) \cap C(V)$$

with $\mathfrak{g}(V)$ and $G(V)$ conform 2.1.

4.2. **Lemma.** *Let V and W be finite dimensional A-modules with partitions $p_V = (m_1, \ldots, m_r)$ and $p_W = (n_1, \ldots, n_s)$. Then*

$$\dim(\operatorname{Hom}_A(V, W)) = \sum_{i=1}^{r} \sum_{j=1}^{s} \min\{m_i, n_j\}$$

Proof. We may assume that V and W are indecomposable A-modules. So $V = A/t^m A$, $W = A/t^n A$, $p_V = (m)$, $p_W = (n)$. It is clear that

$$\dim(\operatorname{Hom}_A(V, W)) = \min\{m, n\}.$$

4.3. **Lemma.** *Let V be a non-degenerate form module with normal form $V = \sum_{i=1}^{r} V_i$, cf.* 3.7. *Then we have*

a) $\dim(Z(V)) = \sum_i \dim(Z(V_i)) + \sum_{i<j} \dim(\operatorname{Hom}_A(V_i, V_j))$.

b) $\dim(\mathfrak{z}(V)) = \sum_i \dim(\mathfrak{z}(V_i)) + \sum_{i<j} \dim(\operatorname{Hom}_A(V_i, V_j))$.

Proof. a) We may assume that $r > 1$. Then V_1 is non-defective so that $V = V_1 \oplus V_1^\perp$ and $V_1^\perp = \sum_{i=2}^{r} V_i$. Put $s = \dim(V_1)$. We consider V_1 as an element of the Grassmann variety $G^s(V)$ of the s-dimensional subspaces of V. Since $C(V)$ is an open subset of the affine space $\mathfrak{c}(V)$, the orbit $\mathcal{O} = C(V)\, V_1$ is an irreducible subvariety of $G^s(V)$. One verifies that

$$\dim(\mathcal{O}) = \dim(\operatorname{Hom}_A(V_1, V_1^\perp)).$$

An element $W \in \mathcal{O}$ is conjugate to V_1 under the action of $Z(V)$ if and only if we have isomorphisms of form modules $W \cong V_1$ and $W^\perp \cong V_1^\perp$. In particular W should be non-defective. The non-defective subspaces form an open and dense subset of $\mathcal{O}$. Since the number of isomorphism classes of form modules of a given dimension is finite there is a unique $Z(V)$-orbit $\mathcal{O}_1$ which is open and dense in $\mathcal{O}$. Condition 3.7(d) implies that $V_1 \in \mathcal{O}_1$. This proves

$$\dim(Z(V)\, V_1) = \dim(\mathcal{O}_1) = \dim(\mathcal{O}).$$

Since the stabilizer of V_1 in the group $Z(V)$ is the product of $Z(V_1)$ and $Z(V_1^\perp)$, it follows that

$$\dim(Z(V)) = \dim(Z(V_1)) + \dim(Z(V_1^\perp)) + \dim(\operatorname{Hom}_A(V_1, V_1^\perp)).$$

Now (a) follows by induction.

b) Consider the subspace $\mathfrak{z}_{ij}$ of the elements $X \in \mathfrak{z}(V)$ with $X V_i \subset V_j$ and $X V_j \subset V_i$ and $X V_p = 0$ for $p \neq i, j$. Assume that $i < j$. Then V_i is non-defective. For $x \in \operatorname{Hom}_A(V_i, V_j)$ there is a unique $x^* \in \operatorname{Hom}_k(V_j, V_i)$ with

$$\beta(x\,v, w) + \beta(v, x^* w) = 0, \qquad v \in V_i,\ w \in V_j$$

One verifies that $x^* \in \operatorname{Hom}_A(V_j, V_i)$ and that $x \mapsto (x, x^*)$ defines a bijective linear mapping

$$\operatorname{Hom}_A(V_i, V_j) \xrightarrow{\sim} \mathfrak{z}_{ij}.$$

The formula follows by linearity.

4.4. Let the function $f\colon \mathbb{N} \to \mathbb{Z}$ be given by

$$f(n)=[\tfrac{1}{2}(n+1)] \quad \text{if } e=0.$$
$$f(n)=[\tfrac{1}{2}n] \quad \text{if } \operatorname{char}(k)\neq 2 \text{ and } e=1.$$
$$f(n)=[\tfrac{1}{2}(n-1)] \quad \text{if } \operatorname{char}(k)=2 \text{ and } e=c=1.$$
$$f(n)=0 \quad \text{if } \operatorname{char}(k)=2 \text{ and } e=1,\ c=0.$$

Theorem. *Let V be a non-degenerate form module with partition $p_V=(m_1,\dots,m_r)$ and index function χ_V. Then*

$$\dim(Z(V))=\sum_{i=1}^{r}(i\,m_i-\chi_V(m_i)).$$

$$\dim(\mathfrak{z}(V))=\sum_{i=1}^{r}(i\,m_i-f(m_i)).$$

Proof. In view of 4.3 and 4.2 and condition 3.7(d) it suffices to consider the indecomposable non-degenerate form modules. So let V be indecomposable with v_1, v_2 as in Table 1. The centralizers $\mathfrak{c}(V)$ and $C(V)$ consist of one-by-one or two-by-two matrices with entries in some quotient ring of A. Using the notation of 3.2 we consider the action of $C(V)$ on the triples (Ψ_1,Φ,Ψ_2) in the space $E_e\oplus E\oplus E_e$. The group $Z(V)$ is the stabiliser of the standard triple chosen in Table 1. The $C(V)$-orbit of the standard triple is denoted by $\mathcal{O}$. We have

$$\dim(Z(V))=\dim(C(V))-\dim(\mathcal{O}).$$

The three cases of 3.2 are treated separately. First assume that $p_V=(m)$ and $\chi_V(m)=l$. The group $C(V)$ consists of the invertible elements of $A/t^m A$. The orbit $\mathcal{O}$ is open in $A_\sigma u_{2l-2+e}$. So we obtain $\dim(Z(V))=m-l$, as required. Assume that $p_V=(m,m)$ and $\chi_V(m)=l$, so that V is $W(m)$ or $W_l(m)$. The group $C(V)$ consists of the invertible 2×2 matrices over $A/t^m A$. The orbit $\mathcal{O}$ is open and dense in

$$A_\sigma u_{2l-2+e}\oplus A t^{1-m}\oplus A_\sigma u_{2l-2+e}.$$

So we have $\dim(\mathcal{O})=m+2l$ and $\dim(Z(V))=3m-2l$. Finally, assume that $p_V=(m,m-1)$, so that V is $D(m)$ with $m\geqq 2$ and $\chi_V=[m;m]$. Now $C(V)$ consists of the invertible 2×2 matrices $g=(g_{ij})$ with $g_{11}\in A/t^m A, g_{12}\in tA/t^m A$ and $g_{21}, g_{22}\in A/t^{m-1}A$. So we have $\dim(C(V))=4m-3$. The orbit $\mathcal{O}$ is open and dense in

$$A_\sigma u_{2m-2}\oplus A t^{2-m}\oplus A_\sigma u_{2m-4}$$

and hence of dimension $3m-2$. So we have $\dim(Z(V))=m-1$, as required.

For the dimension of the infinitesimal centraliser $\mathfrak{z}(V)$ we use essentially the same technique. The Lie algebra $\mathfrak{c}(V)$ is the closure of $C(V)$, so it is obtained from the above description of $C(V)$ by omitting the word "invertible". Using 2.1 and 2.6(2) and 2.5 one verifies that $\mathfrak{z}(V)$ is the kernel in $\mathfrak{c}(V)$ of the linear mapping $P=(P_0,P_1,P_2,P_3)$ given by

$$P_0(X)=\varphi(X v_1, v_2)+\varphi(v_1, X v_2)\in E$$
$$P_i(X)=\pi_e(h^e \varphi(X v_i, v_i))\in E_e \quad \text{for } i=1,2$$
$$P_3(X)=\mathrm{trace}(X)\in k.$$

Here we used the vectors v_1, v_2 chosen above. We leave it to the reader to calculate the dimension of the image of P in the various cases.

4.5. *Remark.* One may use the method of 4.3(a) to analyse the finite group $A(V) = Z(V)/Z(V)^0$ where $Z(V)^0$ is the identity component of $Z(V)$. In this way one shows that $A(V)$ is generated by commuting contributions of the indecomposable components in the normal form. See [6] and [9].

5. The Non-Connected Groups

5.1. In this section we assume that $e=0$ and that V is a non-defective form module. So the group $G=G(V)$ has two connected components. This group is generated by the reflexions r_w with $w\in V$, $\alpha(w)\neq 0$, given by

$$r_w(v)=v-\beta(v,w)\,\alpha(w)^{-1}\,w.$$

The identity component G^0 of G consists of the products of an even number of reflexions. If $\mathrm{char}(k)\neq 2$ then G^0 consists of the elements $g\in G$ with $\det(g)=1$, so that all unipotent elements of G belong to G^0.

5.2. **Proposition.** *Assume* $\mathrm{char}(k)=2$ *and* $c=1$, *so that* $u=1+T$ *is a unipotent element of* G. *Let* n *be the exponent function of the partition* p_V *of the form module* V. *We have* $u\in G^0$ *if and only if* $\sum n(i)$ *is even.*

Proof. We may assume that V is indecomposable. First assume $V\cong W(m)$. Then $p=(m,m)$ so that $\sum n(i)=2$. Let v_1, v_2 be a Jordan basis of V with $\psi(v_1)=\psi(v_2)=0$. For $\lambda\in k$ let u_λ: $V\to V$ be defined by

$$u_\lambda(a_1 v_1+a_2 v_2)=a_1(1+\lambda t)\,v_1+a_2\,\sigma(1+\lambda t)^{-1}\,v_2.$$

One verifies that $u_\lambda\in G$ for all $\lambda\in k$, that $u_1=u$ and $u_0=1$. This proves that $u\in G^0$.

Assume $V\cong V(m)$. Then $p=(m)$ so that $\sum n(i)=1$. Let v_1 be a Jordan basis of V with $\psi(v_1)=u_{2s}$ where $m=2s$. Put $w=t^s v_1$. One verifies that $\alpha(w)=1$. So we have the reflexion r_w and the element $u_1=r_w u$ in G. Put $T_1=u_1-1$. One verifies that

$$T_1\,v=t\,v-\beta((1+t)\,v, t^s v_1)\,t^s v_1.$$

Using $\mathrm{char}(k)=2$ we obtain $T_1\,t^i v_1=t^{i+1} v_1$ for all $i\neq s-1$, and $T_1\,t^{s-1} v_1=0$. So $V_1=(V, T_1)$ is a form module with partition $p(V_1)=(s,s)$. Therefore V_1 is isomorphic to $V(s)\oplus V(s)$ or to $W(s)$. In either case we get $u_1\in G^0$ so that $u\notin G^0$.

5.3. *Remark.* Let $\mathrm{char}(k)=2$, $c=1$ and $V=V(2r)$ with $r\geqq 2$. The group G is semisimple of type D_r, so the variety of the unipotent elements of G^0 has dimension

$\dim(G)-r$, cf. [8] 4.4. By 4.4 above however, the conjugacy class $\mathcal{O}$ of $1+T$ in G satisfies

$$\dim(\mathcal{O})=\dim(G)-r+1.$$

This is an alternative proof that $1+T\notin G^0$.

5.4. **Proposition.** *The G-orbit $\mathcal{O}$ of $c+T$ in G or $\mathfrak{g}$ splits into two G^0-orbits if and only if the index function χ_V of V satisfies $\chi_V(m)\leqq\frac{1}{2}m$ for all $m\in\mathbb{N}$.*

Remark. Table 1 shows that the form modules V with $\chi_V(m)\leqq\frac{1}{2}m$ for all m, are the orthogonal direct sums of the modules $W(m)$ with m even.

Proof. The orbit $\mathcal{O}$ splits into two G^0-orbits if and only if the centraliser $Z(V)$ is contained in G^0. Assume that $\chi_V(m)\leqq\frac{1}{2}m$ for all m. Using the remark one verifies that the subspace

$$F=\sum_n t^n\ker(t^{2n})$$

of V satisfies $F=F^{\perp}$ and $\alpha|F=0$. So F is a maximal totally singular subspace of V. Since $Z(V)$ stabilises F it follows from [2] III 1.6 that $Z(V)\subset G^0$.

Assume that $\chi_V(m)>\frac{1}{2}m$ for some m. It suffices to prove $Z(V)\not\subset G^0$. So we may assume that V is indecomposable. There are four cases.

1. Assume that $V=V(m)$ and that $\mathrm{char}(k)\neq 2$. Then m is odd so that the multiplication $-1\in Z(V)$ satisfies $-1\notin G^0$.

2. Assume that $V=V(m)$ and that $\mathrm{char}(k)=2$. Then $c=1$ and the element $1+T\in Z(V)$ satisfies $1+T\notin G^0$ by 5.2.

3. Assume that $\mathrm{char}(k)=2$, $c=1$ and $V=W(m)$ where m is odd. We can choose a Jordan basis v_1, v_2 of V with $\psi(v_1)=\psi(v_2)=0$ and $\varphi(v_1,v_2)=u_{m-1}$. Put $w_1=v_2$ and $w_2=(1+t)v_1$. One verifies that $\psi(w_1)=\psi(w_2)=0$ and $\varphi(w_1,w_2)=u_{m-1}$. Therefore we may define $u_1\in Z(V)$ by

$$u_1(a_1v_1+a_2v_2)=a_1w_1+a_2w_2 \qquad (a_1,a_2\in A).$$

Clearly $u_1^2=1+T$. Put $T_1=u_1-1$. Then $T_1^2=T$. It follows that (V,T_1) is a form module with partition $p(V,T_1)=(2m)$. By 5.2 we have $u_1\notin G^0$.

4. Assume that $\mathrm{char}(k)=2$, $c=0$ and $p_V=(m,m)$ and $\chi_V=[m;l]$ where $l\geqq\frac{1}{2}(m+1)$. Choose a Jordan basis v_1, v_2 of V with $\psi(v_1)=t^{2-2l}$, $\varphi(v_1,v_2)=t^{1-m}$ and $\psi(v_2)=0$. Put $w_2=v_2+t^{2l-1-m}v_1$. One verifies that $\varphi(v_1,w_2)=\varphi(v_1,v_2)$ and that $\psi(w_2)=\psi(v_2)$. So we may define $u_2\in Z(V)$ by

$$u_2(a_1v_1+a_2v_2)=a_1v_1+a_2w_2 \qquad (a_1,a_2\in A).$$

Put $T_2=u_2-1$. Then (V,T_2) is a form module with partition

$$p(V,T_2)=(2^{2m+1-2l},1^{4l-2m-2})$$

so that $\sum n(i)=2l-1$. This proves that $u_2\notin G^0$.

6. Tables of Orbits and Some Comment

6.1. As an illustration of our results we have made tables of the unipotent orbits in the groups $Sp(4)$, $Sp(6)$, $SO(5)$, $SO(7)$, $SO(8)$, $SO(10)$ and of the nilpotent orbits in their Lie algebras. The first column of each table gives the dimension of the centraliser Z, cf. 1.4(a). This number is the codimension of the orbit in the group or the algebra. The orbit itself is characterised by the symbol, cf. 3.9. We distinguish three cases:

case I. $\mathrm{char}(k) \neq 2$, group or algebra
case II. $\mathrm{char}(k) = 2$, orbit in the group $(c=1)$
case III. $\mathrm{char}(k) = 2$, orbit in the Lie algebra $(c=0)$.

Table 2. $Sp(4)$

dim(Z)	symbol	I	II	III
2	4_2	2	3	4
4	2_1^2	4	6	6
6	2_0^2	–	6	6
6	$2_1\,1_0^2$	6	7	7
10	1_0^4	10	10	10
#		4	5	5

Table 3. $Sp(6)$

dim(Z)	symbol	I	II	III
3	6_3	3	4	6
5	$4_2\,2_1$	5	7	8
7	$4_2\,1_0^2$	7	8	9
7	3_1^2	7	7	9
9	3_0^2	–	–	9
9	2_1^3	9	12	12
11	$2_1^2\,1_0^2$	11	13	13
13	$2_0^2\,1_0^2$	–	13	13
15	$2_1\,1_0^4$	15	16	16
21	1_0^6	21	21	21
#		8	9	10

Table 4. $SO(5)$

dim(Z)	I	II	III
2	5_3	$4_3\,1_1$	$3_3\,2_2$
4	$3_2\,1_1^2$	$2_2^2\,1_1$	$2_2^2\,1_1$
6	$2_1^2\,1_1$	$2_1^2\,1_1$	$2_1^2\,1_1$
6	–	$2_2\,1_1^3$	$2_2\,1_1^3$
10	1^5	1^5	1^5
#	4	5	5

Table 5. $SO(7)$

dim(Z)	I	II	III
3	7_4	$6_4\,1_1$	$4_4\,3_3$
5	$5_3\,1_1^2$	$4_3\,2_2\,1_1$	$3_3^2\,1_1$
7	–	$4_3\,1_1^3$	$3_3\,2_2\,1_1^2$
7	$3_2^2\,1_1$	$3_2^2\,1_1$	$3_2^2\,1_1$
9	$3_2\,2_1^2$	$2_2^3\,1_1$	$2_2^3\,1_1$
11	$3_2\,1^4$	$2_2^2\,1_1^3$	$2_2^2\,1_1^3$
13	$2_1^2\,1_1^3$	$2_1^2\,1_1^3$	$2_1^2\,1_1^3$
15	–	$2_2\,1_1^5$	$2_2\,1_1^5$
21	1_1^7	1_1^7	1_1^7
#	7	9	9

Table 6. $SO(8)$

dim(Z)	I	II	III
4	$7_4\,1_1$	$6_4\,2_2$	4_4^2
6	$5_3\,3_2$	4_3^2	4_3^2
8	$5_3\,1_1^3$	$4_3 2_2\,1_1^2$	$3_3^2\,1_1^2$
(2 ×)8	4_2^2	4_2^2	4_2^2
10	$3_2^2\,1_1^2$	$3_2^2\,1_1^2$	$3_2^2\,1_1^2$
12	$3_2\,2_1^2\,1_1$	2_2^4	2_2^4
16	$3_2\,1_1^5$	$2_2^2\,1_1^4$	$2_2^2\,1_1^4$
(2 ×)16	2_1^4	2_1^4	2_1^4
18	$2_1^2\,1_1^4$	$2_1^2\,1_1^4$	$2_1^2\,1_1^4$
28	1_1^8	1_1^8	1_1^8
#	12	12	12

Table 7 $SO(10)$

dim(Z)	I	II	III
5	$9_5\,1_1$	$8_5\,2_2$	5_5^2
7	$7_4\,3_2$	$6_4\,4_3$	5_4^2
9	$7_4\,1_1^3$	$6_4\,2_2\,1_1^2$	$4_4^2\,1_1^2$
9	5_3^2	5_3^2	5_3^2
11	$5_3\,3_2\,1_1^2$	$4_3^2\,1_1^2$	$4_3^2\,1_1^2$
13	$5_3\,2_2^2\,1_1$	$4_3\,2_2^3$	$3_3^2\,2_2^2$
13	$4_2^2\,1_1^2$	$4_2^2\,1_1^2$	$4_2^2\,1_1^2$
15	$3_2^3\,1_1$	$3_2^2\,2_2^2$	$3_2^2\,2_2^2$
17	$5_3\,1_1^5$	$4_3\,2_2\,1_1^4$	$3_3^2\,1_1^4$
17	$3_2^2\,2_1^2$	$3_2^2\,2_1^2$	$3_2^2\,2_1^2$
19	$3_2^2\,1_1^4$	$3_2^2\,1_1^4$	$3_2^2\,1_1^4$
21	$3_2\,2_1^2\,1_1^3$	$2_2^4\,1_1^2$	$2_2^4\,1_1^2$
25	$2_1^4\,1_1^2$	$2_1^4\,1_1^2$	$2_1^4\,1_1^2$
29	$3_2\,1_1^7$	$2_2^2\,1_1^6$	$2_2^2\,1_1^6$
31	$2_1^2\,1_1^6$	$2_1^2\,1_1^6$	$2_1^2\,1_1^6$
45	1_1^{10}	1_1^{10}	1_1^{10}
#	16	16	16

Table 8. Number of orbits for $SO(n)$, n even

n	I	II	III
4	4	4	4
6	5	5	5
8	12	12	12
10	16	16	16
12	31	32	32
14	43	45	45
16	75	80	80
18	105	115	115
20	168	187	188
22	236	268	270

In the columns marked by I, II, III it is indicated whether an orbit occurs in the given case. For the symplectic groups we give in this columns also the dimension of the infinitesimal centraliser $\mathfrak{z}$. For the special orthogonal groups the dimensions of $\mathfrak{z}$ are not given. The bottom line of each table gives the number of orbits in the three cases. We have treated the special orthogonal groups $SO(n)$ instead of $O(n)$. This reduces the number of orbits in case II if n is even, cf. 5.2. On the other hand some orbits split, cf. 5.4. In the table of $SO(8)$ this splitting occurs twice.

6.2. For the special orthogonal groups orbits with different symbols are arranged in lines according to codimension and related structure. In fact for a connected reductive group over a field of arbitrary characteristic there is a natural correspondence between its unipotent classes and the nilpotent classes in its Lie algebra. This correspondence is given by the condition that the closure of the nilpotent class is an irreducible component of the tangent cone in the unit element of the closure of the unipotent class. This correspondence seems to be a bijective map for $SO(n)$ with $n \leqq 12$ or $n = 14$, 16, 18.

In these cases one might hope for the existence of a G-equivariant bijective morphisme $f\colon N \to U$ where N is the set of the nilpotent elements of $\mathfrak{g}$ and U is the set of the unipotent elements of G. Such a mapping however cannot exist for $SO(n)$ if $\mathrm{char}(k) = 2$. In fact, let T be a regular nilpotent element of N, so that $u = f(T)$ is a regular element of U, cf. [8]. Then $l\colon \mathbb{A}^1 \to Z(u)$ given by $l(x) = f(x\,T)$ is a path connecting u with the unit element inside the centraliser $Z(u)$. This contradicts [8] (4.12). This argument is due to Springer.

If $\mathrm{char}(k)=2$ and $n=13$, 15, 17 or $n\geqq 19$, there are more nilpotent classes in $\mathfrak{so}(n)$ than unipotent classes in $SO(n)$. In Table 8 we give the number of orbits in the three cases for $SO(n)$ with n even and less than 23.

6.3. In case II the tables for $SO(2l+1)$ and $Sp(2l)$ are isomorphic. This is due to the bijective inseparable isogeny $SO(V)\to Sp(V/V^{\perp})$ which exists for a defective but non-degenerate form space V.

The triality automorphism of Spin(8) does not survive on the level of $SO(8)$ in characteristic two. One can show this using the three orbits of codimension 8 with infinitesimal centralisers of dimension 8,8 and 10 respectively.

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